Vectors and Points

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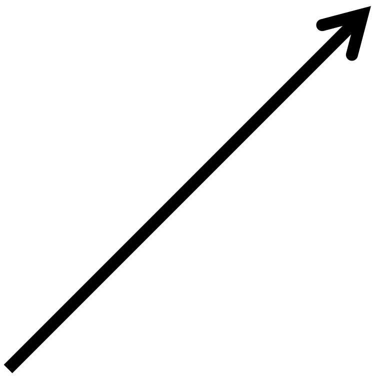
Linear Space

- A *linear space* (or *vector space*) is a set whose elements are called *vectors*
- A *geometric vector* is a vector that has direction and lenght, useful for visualization



Linear Combination

- A *linear combination* of vectors in a linear space is defined as

$$v = a_1 v_1 + ... + a_n v_n, \forall a_i$$

- All possible linear combinations in a linear space forms a set called the *span* of a linear space (or *linear span* or *linear hull*)

Linear Dependence

- A set S is *linearly dependent* if one of its vectors can be defined as a unique non-zero linear combination of the others

$$v_i = a_1 v_1 + ... + a_{i-1} v_{i-1} + ... + a_{i+1} v_{i+1} + ... + a_n v_n$$

- Otherwise the set is *linearly independent*

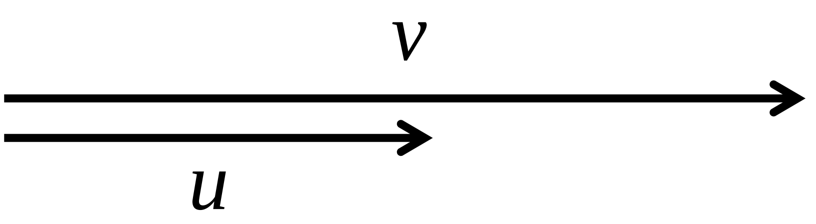
Linear Dependence

- Two colinear vectors

$$S = \{0, u, v\}$$

$$u = \frac{1}{2} v$$

$$v = 2u$$



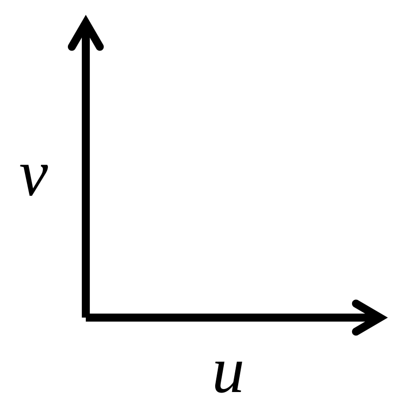
Linear Independence

- Two orthogonal vectors
- There is no non-zero scalar that when multiplied with a vector in *S* produces another vector in *S*

$$S = \{0, u, v\}$$

$$u \neq av$$

$$v \neq au$$



Linear Basis

- A linear basis B is a linearly independent set of n vectors
- For every vector v in B there is one linear combination that equals v

$$v = a_1 v_1 + ... + a_n v_n$$

- Defining a fixed-basis the scalars $(a_1, ..., a_n)$ can identify v

Linear Map

- Map between two linear spaces V and W
  $f : V \to W$ where for all v in V and all scalars a

  1. $f(v_1 + v_2) = f(v_1) + f(v_2)$
  2. $f(av) = af(v)$
- More directly

$$f(av_1 + v_2) = af(v_1) + f(v_2)$$

Linear Map

Linear

$$f(x) = 5x$$

$$f(ax + y) = 5(ax + y)$$

$$= 5ax + 5y$$

$$= af(x) + f(y)$$

Non-Linear

$$1.\ f(x, y) = x^2 y$$

$$f(2x, 2y) = 2x^2 2y$$

$$\neq 2f(x, y) = 2x^2 y$$

$$2.\ f(x) = 2x \underbrace{+1}$$

$$f(2x) = 4x + 1$$

$$\neq 2f(x) = 4x + 2$$

Linear Map

- Mapping a vector *v* from *V* to *W* is mapping a linear basis for *V* from *V* to *W*, not forming another linear basis for *W*

$$f(v) = f(a_1 v_1 + \ldots + a_n v_n)$$
$$= a_1 f(v_1) + \ldots + a_n f(v_n)$$

Linear Map

- By storing a basis for *V* already mapped to *W* in the columns of a matrix we can express *f(v)* in terms of the basis for *W* as a linear combination

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Column Space

- The linear space spanned by the column vectors

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$S = \{0, \begin{bmatrix} 2 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 2 \end{bmatrix}^T\}$$

Row Space

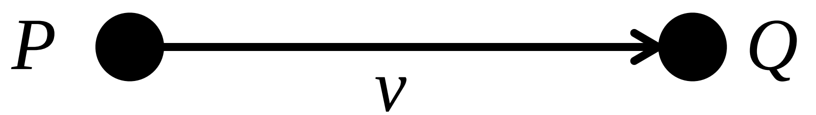
- The linear space spanned by the row vectors

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$S = \{0, \begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \end{bmatrix}\}$$

Affine Space

- An *affine space* is a set of points W and a linear space V
- The relation between W and V is that for every pair of points (P, Q) in W there is a vector $v = Q - P$ in V
- Accordingly, $Q = P + v$ and $P = Q - v$



Affine Combination

- An *affine combination* of points in an affine space is a linear combination defined as

$$P = a_1 P_1 + ... + a_n P_n, \sum_{i=1}^{n} a_i = 1$$

- All possible affine combinations in an affine space forms a set called the *span* of an affine space (or *affine span* or *affine hull*)

Affine Dependence

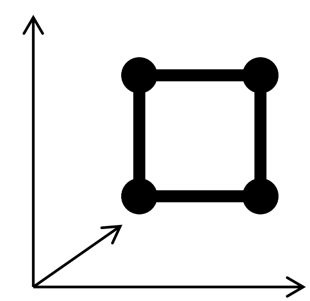
- A set is *affinely dependent* if one of its points can be defined as a unique non-zero linear combination of the others

$$P_i = a_1 P_1 + \ldots + a_{i-1} P_{i-1} + \ldots + a_{i+1} P_{i+1} + \ldots + a_n P_n, \sum_{i=1}^{n} a_i = 1$$

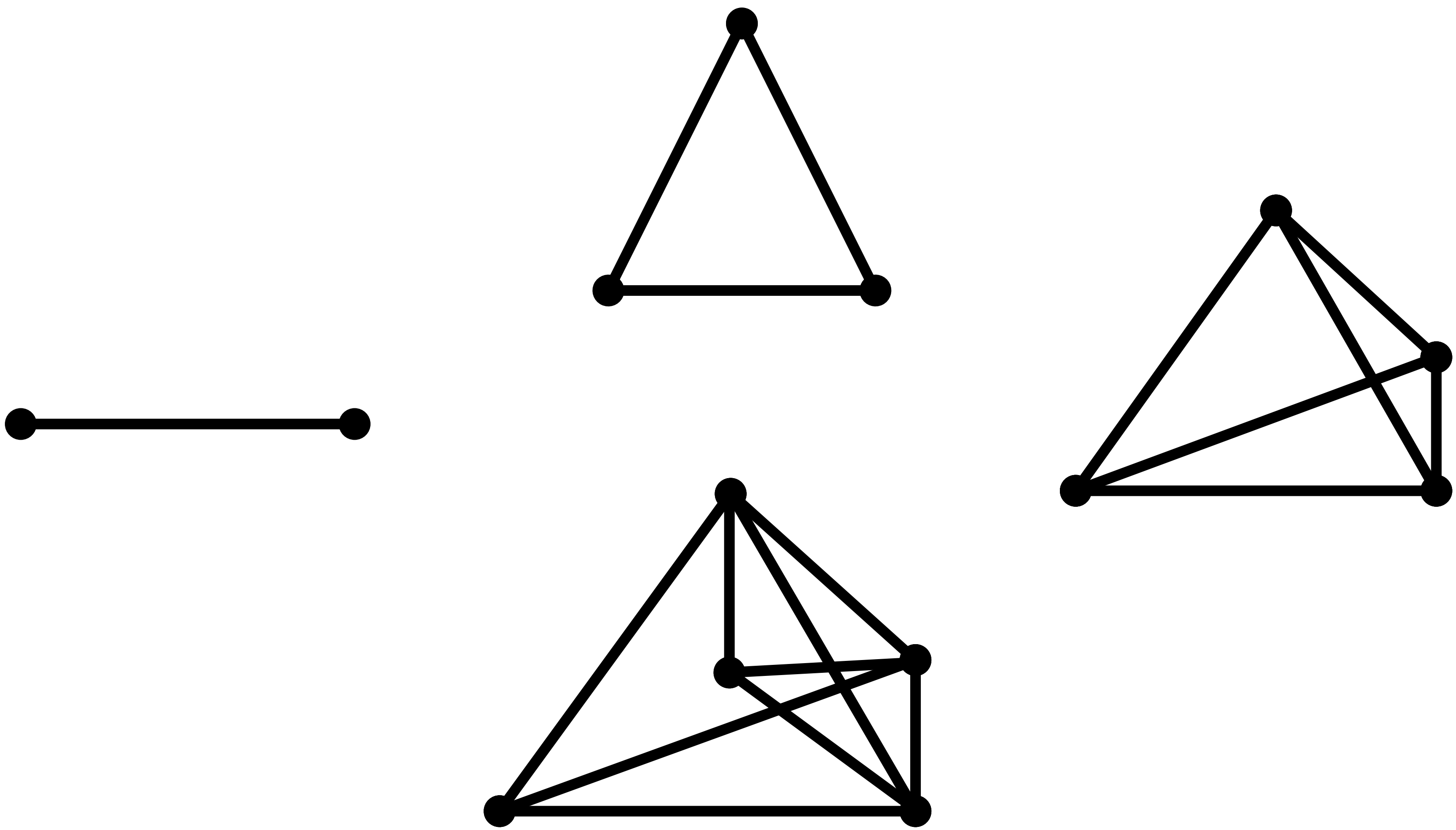
- Otherwise the set is *affinely independent* and the scalars are called *barycentric coordinates*

Affine Dependence

- Two coincident points
- Three colinear points
- Four coplanar points



Affine Independence



Affine Frame

- Defining a fixed-point *O* for *W* and a fixed-basis *B* for *V* any point in *W* can be defined as

$$P = O + v = O + a_1 v_1 + ... + a_n v_n$$

- The scalars $(a_1, ..., a_n)$ can identify *v*
- An affine frame is a point *O* and a basis *B*

Affine Map

- Map a point *P* from an affine space *A* with origin O_A and basis $\{v_1,\dots,v_n\}$ to an affine space *B* with origin O_B and basis $\{w_1,\dots,w_m\}$

$$f(P) = f(O_A) + x_1 f(v_1) + \dots + x_n f(v_n)$$

$$f(v_j) = a_1 w_1 + \dots + a_m w_m$$

$$f(O_A) = y_1 w_1 + \dots + y_m w_m$$

Affine Map

- A linear map followed by a vector add

$$Ax + y$$

$$\begin{bmatrix} A & y \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + y \\ 1 \end{bmatrix}$$

Standard Cartesian Frame

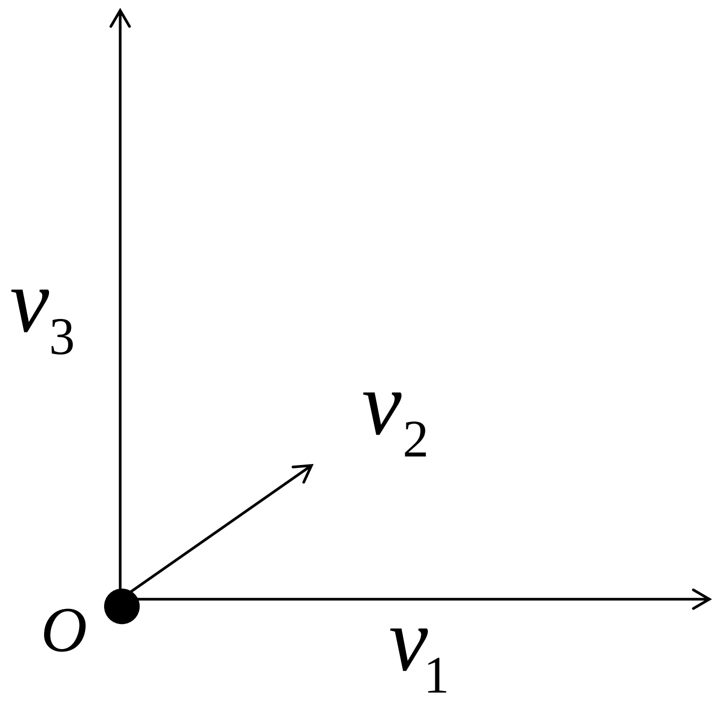
- A fixed-point $O = (0,0,0)$
- A 3-dimensional basis $\{v_1, v_2, v_3\}$ where

$v_1 = (1,0,0), v_2 = (0,0,1), v_3 = (0,1,0)$

$$P = O + a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3$$

Standard Cartesian Frame



Standard Cartesian Map

$$f(P) = f(O_A) + x_1 f(v_1) + x_2 f(v_2) + x_3 f(v_3)$$

$$f(v_1) = (a_1, 0, 0),\ f(v_2) = (0, 0, a_2),\ f(v_3) = (0, a_3, 0)$$

$$f(O_A) = O_B$$

References

- http://www.essentialmath.com/references
- Jim Van Verth - Essential Math for Games Programmers and Interactive Applications, 2nd Edition

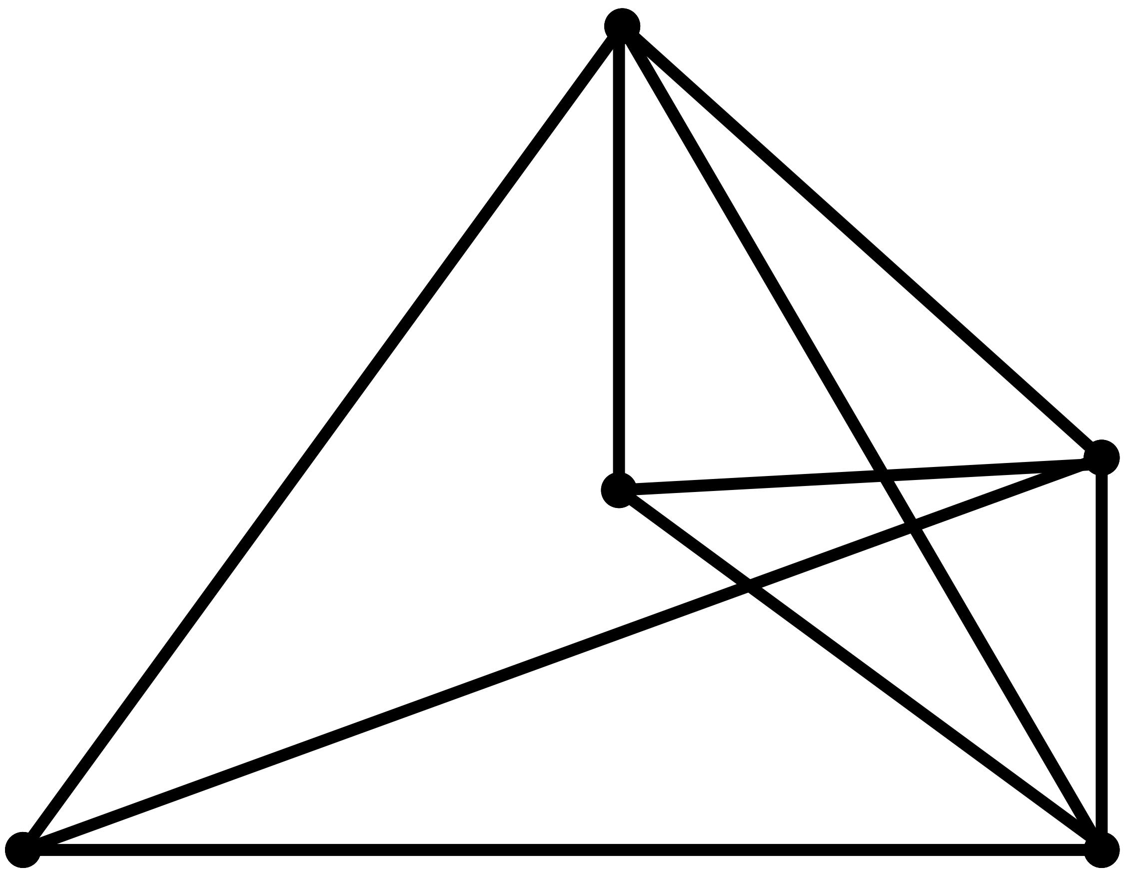
Appendix
Convex Combination

- A *convex combination* of a set of points is defined as

$$P = a_1 P_1 + ... + a_n P_n, a_i \geq 0, \sum_{i=1}^{n} a_i = 1$$

- All possible convex combinations of a convex set is called the *span* of the convex set (or *convex span* or *convex hull*)

Appendix
Convex Dependence



Appendix
Convex Independence