Irlan Robson
dnairlan@gmail.com

Distance Constraint

Assuming we have a 2 body system, let x be the position vector, and

$$\dot{x} \in \mathbb{R}^{12x1} = V = \begin{pmatrix} v_1 \\ \omega_1 \\ v_2 \\ \omega_2 \end{pmatrix} \tag{1}$$

be the velocity vector where v and ω are the linear and the angular velocity, respectively. The system' block diagonal mass matrix collecting the masses and 3-by-3 inertia tensors is defined as

$$M \in \mathbb{R}^{12x12} = \begin{pmatrix} M_1 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & M_2 & 0 \\ 0 & 0 & 0 & I_2 \end{pmatrix} \tag{2}$$

The inverse matrix is simply

$$M^{-1} \in \mathbb{R}^{12x12} = \begin{pmatrix} M_1^{-1} & 0 & 0 & 0 \\ 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & M_2^{-1} & 0 \\ 0 & 0 & 0 & I_2^{-1} \end{pmatrix} \tag{3}$$

The equation of motion is then

$$\dot{V} = M^{-1}(F_E + F_C) \tag{4}$$

where $F_E \in \mathbb{R}^{12x1}$ and $F_C \in \mathbb{R}^{12x1}$ are the external and constraint forces, respectively. Note that each entry in M^{-1} is written here as a 3-by-3 matrix. Using block matrices will simplify further computations.

The distance constraint is an implicit scalar function of the position which by itself is a function of time. It keeps two points fixed on the bodies separated by a distance L .

$$C = \|d\| - L \tag{5}$$

where $d = p_2 - p_1$ and p_1 and p_2 are the points.

Diferentiating equation 5 with respect to time using the chain rule of differentiation gives the velocity constraint.

$$\dot{C} = n^T (v_2 + w_2 \times r_2 - v_1 - w_1 \times r_1) \tag{6}$$

where $n = \frac{d}{\|d\|}$. Generally, the first term is the *s*-by-*n* Jacobian in the number of constraints and bodies respectively. The Jacobian is found explicitly by separating the velocities of the bodies from the other terms using the scalar triple product cyclic identity and algebraic manipulation.

$$J \in \mathbb{R}^{1x12} = \begin{pmatrix} -n^T & -(r_1 \times n)^T & n^T & (r_2 \times n)^T \end{pmatrix} \tag{7}$$

The velocity constraint is satisfied when

$$JV \in \mathbb{R}^{1x1} = 0 \tag{8}$$

which is true because the Jacobian is linear in terms of the velocity. Notice that the Jacobian is a row vector of length 12.

The virtual work principle states that constraint forces must do no work. They're force laws designed by us in order to keep the system in a desired state, and should not modify the energy of the system. Therefore, it is implied that the direction of a constraint force must be orthogonal to all legal displacements, that is,

$$F_C = J^T \lambda \tag{9}$$

where λ is the magnitude of the constraint forces, a column vector of length 12, also known as lagrange multipliers. We must solve for the multipliers so we can step the simulation using equation 4.

By using an iterative solution such as the popular Sequential Impulses technique, three steps can be used for solving for λ .

1. Apply external forces.
2. Apply constraint impulses.
3. Apply constrained velocities.

In step 1 we integrate the velocities using the external forces, so that the constraint forces are allowed to fight simultaneously with the external forces.

$$V^2 = V^1 + h M^{-1} F_E \tag{10}$$

Here the superscript means the time step.

Next we solve for λ in terms of a velocity change using the impulse law in Euler form

$$P_C = M(V_2 - V_1) \Leftrightarrow V_2 = V_1 + M^{-1} J^T \lambda \tag{11}$$

where the subscript means the velocity before and after the constraint impulse. Plugging in V_2 in equation 8 it is possible to solve for λ .

$$\lambda = (J M^{-1} J^T)^{-1} - J V_1 \tag{12}$$

Once we have the velocities we integrate the positions so we have new initial conditions for sucessive integrations.

Notice that constraint stabilization is not taken in consideration here for simplicity, but it can (and should) be added using i.e. Baumgarte's method or position projection.

The rest of this paper works out step-by-step in the derivation of λ for the distance constraint. Using block matrices and their identities the derivation workload is reduced.

Let $L_1 = -n,\ A_1 = -(r_1 \times n),\ L_2 = n,\ A_2 = r_2 \times n$ be column vectors of length 3.

$$J \in \mathbb{R}^{1x12} = \begin{pmatrix} L_1^T & A_1^T & L_2^T & A_2^T \end{pmatrix} \tag{13}$$

$$J^T \in \mathbb{R}^{12x1} = \begin{pmatrix} L_1 \\ A_1 \\ L_2 \\ A_2 \end{pmatrix} \tag{14}$$

$$J M^{-1} \in \mathbb{R}^{1x12} = \begin{pmatrix} L_1^T M_1^{-1} & A_1^T I_1^{-1} & L_2^T M_2^{-1} & A_2^T I_2^{-1} \end{pmatrix} \tag{15}$$

$$J M^{-1} J^T \in \mathbb{R}^{1x1} = \left(L_1^T M_1^{-1} L_1 + A_1^T I_1^{-1} A_1 + L_2^T M_2^{-1} L_2 + A_2^T I_2^{-1} A_2 \right) \tag{16}$$

$$J V \in \mathbb{R}^{1x1} = \left(L_1^T v_1 + A_1^T \omega_1 + L_2^T v_2 + A_2^T \omega_2 \right) \tag{17}$$

$$\lambda \in \mathbb{R}^{1x1} = (J M^{-1} J^T)^1 - J V \tag{18}$$

$$J^T \lambda \in \mathbb{R}^{12\text{x}1} = \begin{pmatrix} L_1 \lambda \\ A_1 \lambda \\ L_2 \lambda \\ A_2 \lambda \end{pmatrix} \tag{19}$$